Cluster-weighted modeling: Estimation of the Lyapunov spectrum in driven systems

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Cluster-weighted modeling based techniques are shown to be accurate, efficient, and robust in application to the problem of computing the Lyapunov spectrum from time-series data. We develop a method that is appropriate for application to driven nonlinear dynamical systems and show, in particular, that it is possible to estimate both global and local Lyapunov exponents through this technique. For dynamics on strange nonchaotic attractors, the present approach correctly determines a largest Lyapunov exponent that is negative.

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I. INTRODUCTION

An ongoing theme in nonlinear science research since the late 1970s has been the analysis of time-series data to extract information about the underlying dynamical system [1–4]. The ubiquity of nonlinearity, combined with the impossibility of monitoring all variables in any but the simplest experimental systems, has made this a subject of considerable importance. Reconstruction of the phase-space dynamics from monitoring a single dynamical variable can be carried out by techniques [5] that are based on the Takens embedding theorem [6]. Such reconstruction reveals underlying attractors, and their characterization can be carried out through the computation of metric, dynamical, and topological quantities that are invariant under coordinate transformations. Metric invariants [7,8] such as fractal dimensions or multifractal measures involve statistical convergence of the distribution of points in phase space, while dynamical invariants like the Lyapunov exponents [9] are based on the evolution properties of the trajectories. Topological invariants are derived from the stretching and folding mechanisms present in the dynamics. Together these different quantities provide robust characterization of low-dimensional chaotic dynamics [10,11].

In the present paper, we develop a different approach to the computation of dynamical invariants. From a given time series, we construct local nonlinear models of the dynamics from which a global model can be synthesized. The local models are probabilistic and since they make no assumptions about the underlying dynamical system, this method provides an unbiased technique for inferring phase-space properties.

One motivation for the present work comes from the study of driven dynamical systems. Standard techniques for dynamics reconstruction [12] have been devised for autonomous systems and when the underlying dynamics is forced, these can give conflicting or erroneous results [13]. Although Takens' embedding theorem can be reformulated for application to forced systems (with minor restrictions on the nature of the forcing) [14], it has not been widely utilized. As a practical matter, it is frequently unknown whether an autonomous or nonautonomous process gives rise to the time-series data that are measured in an experiment. Indeed, many natural systems that merit study within this framework, such as the weather or the climate, or medical time series, are in fact driven dynamical systems.

The framework of *cluster-weighted modeling* can be fruitfully applied to the analysis of time-series data from driven dynamical systems. Cluster-weighted models (CWM's) are supervised learning techniques which are based on the joint probability density estimation of a set of input and output (target) data. They have been used in characterization of low-dimensional dynamical systems such as the Lorenz attractor, and in the analysis of highly nonlinear time-series data monitored from musical instruments [15]. The superior predictive properties of CWM's allow for accurate estimation of Lyapunov exponents on both chaotic and nonchaotic attractors. Furthermore, as a modeling strategy, CWM's offer distinct advantages since all available information can be incorporated within local models which can be defined with as much flexibility as desired.

There are other nonparametric predictive models that have been developed [16,17] and have been successfully applied in many cases. Although these have been shown to be capable of predicting low-dimensional chaotic dynamics, their utility in the analysis of intermittent motion can be limited due to the occurrence of bursts with rare-event statistics [18]. Further, these methods are not ideally suited for application to forced systems, and in this respect, the present CWM method should be viewed as a complementary technique.

The necessity of accurate methods that are applicable for driven dynamical systems arises additionally from the fact that with forcing, it is possible to have aperiodic but nonchaotic behavior. Studies of quasiperiodically forced nonlinear systems have shown that there can be strange nonchaotic attractors (SNA's) with *negative* largest Lyapunov exponent [19]. However, the computation of the local Jacobian by estimating the divergence of nearby trajectories in phase space [12,20] or in the tangent space [21] has been shown to yield incorrect Lyapunov exponents. Indeed, a recent study,

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wherein different time-series methods for estimating Lyapunov exponents were compared, arrived at the conclusion that detection of nonchaotic dynamics from experimental time series is numerically impossible [13].

The present CWM formalism, which overcomes this shortcoming, is discussed in detail in Sec. III. To set the context, however, in the next section we briefly recapitulate the standard methodology for obtaining Lyapunov exponents from trajectories or from time series. Representative applications are presented in Sec. IV, and the paper concludes with a summary in Sec. V.

II. LYAPUNOV EXPONENTS

The Lyapunov exponents of a dynamical system are important in understanding stability properties. They can be determined by observing the evolution of small deviations of a fiducial orbit in phase space [21]. When the dynamics is known, the formalism for extracting the spectrum of Lyapunov exponents is straightforward. Consider a discrete dynamical system where the orbit $\mathbf{z}(n) \in \mathbb{R}^d$ is determined by a mapping,

$$\mathbf{z}(n+1) = \mathbf{f}(\mathbf{z}(n)). \tag{1}$$

The evolution of a perturbation $\delta\mathbf{z}(n)$ is governed by the $n \times n$ Jacobian matrix $D\mathbf{f}(n)$,

$$\delta\mathbf{z}(n+1) = D\mathbf{f}(\mathbf{z}(n))\,\delta\mathbf{z}(n) \equiv D\mathbf{f}(n)\,\delta\mathbf{z}(n). \tag{2}$$

In N steps, the deviation expands by the factor

$$D\mathbf{f}^N = D\mathbf{f}(N)\; D\mathbf{f}(N-1) \ldots D\mathbf{f}(1) \tag{3}$$

and the Lyapunov exponents are the logarithms of the eigenvalues of the matrix

$$\mathcal{L} = \lim_{N\to\infty} [(D\mathbf{f}^N)^\dagger (D\mathbf{f}^N)]^{1/2N}. \tag{4}$$

In practice, the method of QR decomposition [22] gives superior numerical stability. The Jacobian at each time step is decomposed into an orthogonal matrix $Q(n)$ and an upper triangular matrix $R(n)$ as

$$D\mathbf{f}(n+1)\; Q(n) = Q(n+1)\; R(n+1) \tag{5}$$

with $Q(0)=I$. The Lyapunov exponents are given by

$$\Lambda_i = \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \ln R_{ii}(n), \quad i = 1, \ldots, d, \tag{6}$$

d being the dimension of the phase space. Finite-time Lyapunov exponents can be evaluated in an analogous manner; the k-step exponent is

$$\Lambda_i(k) = \frac{1}{k}\sum_{n=1}^{k} \ln R_{ii}(n), \tag{7}$$

but unlike the asymptotic exponent, the finite-time Lyapunov exponent depends on initial conditions. The corresponding probability density,

$$P_i(\Lambda,k)d\Lambda = [\text{probability that } \Lambda_i(k) \text{ takes a value between } \Lambda \text{ and } \Lambda + d\Lambda] \tag{8}$$

is, however, stationary.

It is crucial that an accurate representation of the Jacobian be obtained in order to get reliable Lyapunov exponents from experimental time-series data. For phase-space reconstruction [5], the Takens method uses a scalar time series $s(n)$ and recreates vectors in a d-dimensional Euclidean space as

$$\mathbf{x}(n) = \{s(n), s(n+T), \ldots, s(n+(d-1)T)\} \tag{9}$$

where T is a suitably chosen time delay obtained from analysis (say of the mutual information [23]) of the time series [21]. The standard algorithms to obtain Lyapunov exponents from the reconstructed phase-space dynamics have been discussed extensively in the literature [12,20]. Here we intend to construct accurate local models for the dynamics from which the Jacobian can be obtained, and the methodology of CWM which achieves this objective is discussed in the following section.

III. CLUSTER-WEIGHTED MODELING

Consider that the underlying dynamical system provides a model $\mathcal{M}$ which outputs the time series $\mathbf{y}$ given input data $\mathbf{x}$. The essential idea underlying the CWM procedure is to approximate this unknown model by a set of local nonlinear models denoted $\mathcal{C}_k, k=1,2,\ldots,K$. The global model $\mathcal{M}$ is then constructed as a Gaussian mixture over a suitably chosen set $\{\mathcal{C}\}$.

Each local model $\mathcal{C}_k$ is obtained via fitting a nonlinear function, namely, by obtaining a set of parameters β_k in suitably chosen nonlinear maps

$$\mathbf{y} = \mathbf{f}(\mathbf{x}, \beta_k). \tag{10}$$

For generality, the maps $\mathbf{f}$ are taken to be polynomial functions. The joint probability distribution $p(\mathbf{y},\mathbf{x})$ is expressed as a sum over the densities coming from each local model,

$$p(\mathbf{y},\mathbf{x}) = \sum_{k=1}^{K} p(\mathbf{y},\mathbf{x},\mathcal{C}_k) = \sum_{k=1}^{K} p(\mathbf{y}|\mathbf{x},\mathcal{C}_k)p(\mathbf{x}|\mathcal{C}_k)p(\mathcal{C}_k). \tag{11}$$

The probability of a given local model is denoted $p(\mathcal{C}_k)$, with the usual normalization $\Sigma_{k=1}^{K} p(\mathcal{C}_k)=1$. Denote by $\mathbf{P}_{\mathbf{x},k}$ and $\mathbf{P}_{\mathbf{y},k}$ the covariance matrices for the input and output data, and by D_x and D_y the input and output dimensions. A useful choice for the input distribution $p(\mathbf{x}|\mathcal{C}_k)$ is a Gaussian density,

$$p(\mathbf{x}|\mathcal{C}_k) = \frac{|\mathbf{P}_{\mathbf{x},k}^{-1}|^{1/2}}{(2\pi)^{D_x/2}} \exp[-(\mathbf{x}-\mu_k)^T \cdot \mathbf{P}_{\mathbf{x},k}^{-1} \cdot (\mathbf{x}-\mu_k)]/2 \tag{12}$$

where μ_k are the cluster means. Since the input and output data are related via Eq. (10), this gives an output distribution of the form

$$p(\mathbf{y}|\mathbf{x},\mathcal{C}_k) = \frac{|\mathbf{P}_{y,k}^{-1}|^{1/2}}{(2\pi)^{D_y/2}} \times \exp\{-[\mathbf{y}-\mathbf{f}(\mathbf{x},\beta_k)]^T \cdot \mathbf{P}_{\mathbf{y},k}^{-1} \cdot [\mathbf{y}-\mathbf{f}(\mathbf{x},\beta_k)]\}/2. \tag{13}$$

The posterior probability, which can be computed through Eqs. (11), (12), and (13) as

$$p(\mathcal{C}_k|\mathbf{y},\mathbf{x}) = \frac{p(\mathbf{y},\mathbf{x}|\mathcal{C}_k)p(\mathcal{C}_k)}{\sum_{j=1}^{K} p(\mathbf{y},\mathbf{x}|\mathcal{C}_j)p(\mathcal{C}_j)}, \tag{14}$$

is maximized with respect to model parameters $\beta_\mathbf{k}$ which can be recursively estimated by an iterative search procedure, the expectation-maximization algorithm [24]. New cluster probabilities are estimated from the posterior probability in the usual manner, by

$$p(\mathcal{C}_k) = \frac{1}{N}\sum_{n=1}^{N} p(\mathcal{C}_k|\mathbf{y}_n,\mathbf{x}_n), \tag{15}$$

while the cluster means are updated as

$$\mu_k = \frac{\sum_{n=1}^{N} \mathbf{x}_n p(\mathcal{C}_k|\mathbf{y}_n,\mathbf{x}_n)}{\sum_{n=1}^{N} p(\mathcal{C}_k|\mathbf{y}_n,\mathbf{x}_n)}, \tag{16}$$

and hence the input covariance matrix elements are estimated as

$$\mathbf{P}_{\mathbf{x},k}(i,j) = \langle(\mathbf{x}_i - \mu_i)\cdot(\mathbf{x}_j - \mu_j)\rangle. \tag{17}$$

The optimal model parameters β_k can be determined by the stationarity condition,

$$\sum_{n=1}^{N} \frac{\partial}{\partial \beta_k} \ln p(\mathbf{y}_n,\mathbf{x}_n) = 0, \quad k=1,2,\ldots,K. \tag{18}$$

Thus all the model parameters may be evaluated from

$$\beta_k = \mathbf{B}_k^{-1}\,\mathbf{C}_k \tag{19}$$

where

$$\mathbf{B}_k(i,j) = \langle f_i(\mathbf{x},\beta_k)\cdot f_j(\mathbf{x},\beta_k)\rangle, \tag{20}$$

$$\mathbf{C}_k(i,j) = \langle f_i(\mathbf{x},\beta_k)\cdot y_j\rangle. \tag{21}$$

Finally, the corresponding output covariance matrix $\mathbf{P}_{\mathbf{y},k}$ associated with each cluster is evaluated from the estimated values of the model parameters as

$$\mathbf{P}_{\mathbf{y},k} = \langle[\mathbf{y}-\mathbf{f}(\mathbf{x},\beta_k)]\cdot[\mathbf{y}-\mathbf{f}(\mathbf{x},\beta_k)]^T\rangle. \tag{22}$$

The set of parameters is recursively estimated until they converge to stationary values, and from the local models, the global model $\mathcal{M}$ is constructed. Any desired quantity such as the Lyapunov exponents, correlation functions, or fractal dimension can be computed directly from the global model.

IV. APPLICATIONS AND RESULTS

We apply the CWM methodology to compute the Lyapunov spectra for several model systems both for purposes of validation as well as to demonstrate the suitability of this technique for examining signals that originate from driven dynamics.

Consider, for simplicity, a discrete driven one-dimensional mapping of the general form

$$\phi_{n+1} = F(\phi_n, h(n)), \tag{23}$$

where the forcing is effected through the function $h(n)$ which has dependence on the time index n. In a standard manner, by increasing the phase-space dimension this can be rewritten as an autonomous skew product of the form

$$\phi_{n+1} = f(\phi_n, \theta_n), \tag{24}$$

$$\theta_{n+1} = g(\theta_n), \tag{25}$$

where the latter equation depends on the nature of the forcing. The dynamics of the system is thus characterized by the maps f,g, while the measurements give the observed quantity $s(n)=s(\phi_n)$.

As an example here we consider the case

$$\theta_n = n\omega + \theta_0 \bmod 1, \tag{26}$$

namely, $g(\theta_n)=\theta_n+\omega \bmod 1$. When θ_n appears as the argument of the forcing function, this can be either periodic or quasiperiodic in time according as ω is a rational or irrational number. In the latter case, the dynamics can be on SNA's [19,25]. As indicated earlier, existing methods of Wolf *et al.* [12], Kantz [20], and Brown *et al.* [21] which approximate the Jacobian in Eq. (3) as a local neighborhood map and calculate local divergence have been demonstrated to estimate incorrect Lyapunov exponents for driven systems [13].

The procedure for dynamics reconstruction from a measured times series using the present methodology is as follows.

(1) The CWM algorithm considers $\mathbf{x}(n)$ and $\mathbf{y}(n)\equiv\mathbf{x}(n+1)$ for $n=1,\ldots,1000$ as the input and output data sets. Each local model Eq. (10) is chosen to be a quadratic function in $\mathbf{x}$.

(2) The input distribution and output distribution are evaluated via Eqs. (12) and (13) and hence the posterior probability is obtained from Eq. (14).

(3) The log-likelihood of the data is maximized and the converged parameter values β_k are estimated by Eq. (19) as described.

(4) Since the functional form of the fitting function is known, the Jacobian $D\mathbf{f}$ is readily estimated. A proper choice of the number of clusters required depends on the eigenvalues of the local covariance matrices $\mathbf{P}_{\mathbf{x},k}$ and $\mathbf{P}_{\mathbf{y},k}$ [15]. We have found typically that $K=2$ suffices for prediction and considering a larger number of clusters does not significantly improve results.

(5) Lyapunov exponents are evaluated from Eq. (6).

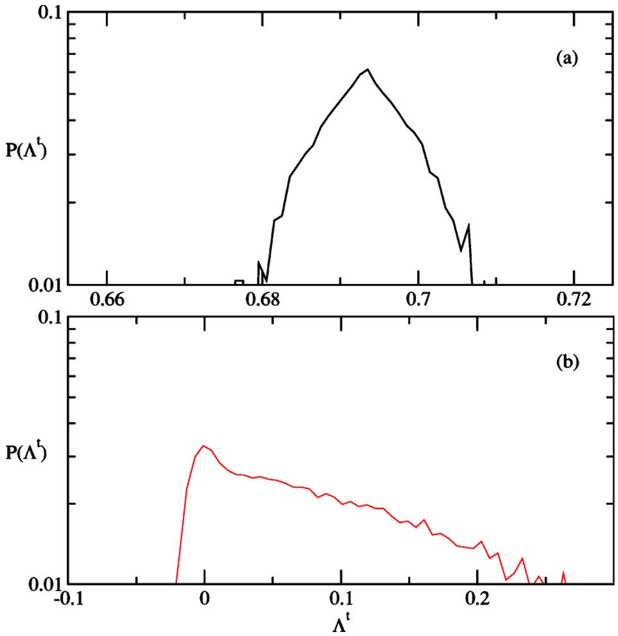

FIG. 1. Characteristic probability distribution of finite-time Lyapunov exponents for (a) fully developed chaos ($\alpha=4$) and (b) type-I intermittency ($\alpha=\sqrt{8}+1-10^{-6}$) in the logistic map. The distributions are for $k=10$, and are obtained from 2^{12} samples.

A. The logistic map

Consider the logistic map $\phi_{n+1}=\alpha\phi_n(1-\phi_n)$, for which the Lyapunov exponent at $\alpha=4$ is $\Lambda=\ln 2$ (the subscript is dropped to simplify notation since there is only one exponent). Taking the input signal $s(n)$ as ϕ_n, one obtains the embedded data $\mathbf{x}(n)$ in a d-dimensional reconstructed phase space via Eq. (9) for the choice of delay time $T=1$. To obtain robust estimates, the above algorithm is applied successively to data segments until the estimated Lyapunov exponents show convergence with iterations. We find $\Lambda^t \approx \Lambda^m = 0.6931$, in good agreement with the exact value; further, increasing the embedding dimension does not change this value.

The power of cluster-weighted modeling is further shown in its ability to accurately predict local dynamics, and from this to estimate the finite-time Lyapunov exponent distribution. It is known that this quantity has a characteristic dependence on the nature of the dynamics [26]. Shown in Fig. 1 is the probability density for the finite-time Lyapunov exponents $P(\Lambda, k=10)$, computed from time series from the logistic map for fully developed chaos ($\alpha=4$), and intermittent chaos ($\alpha=1+\sqrt{8}-10^{-6}$). In both these cases, the present technique correctly obtains the density, giving a cusp around $\Lambda^t \approx 0.693$ in the former case, Fig. 1(a), and an asymmetric fat-tailed distribution in the latter [18,26], Fig. 1(b). The fidelity with which the local dynamics can be reconstructed is a crucial feature that contributes to the success of CWM in estimating Lyapunov exponents for forced systems.

B. Quasiperiodic forcing

The quasiperiodically forced logistic map and the forced Hénon map are discrete driven systems where, depending on parameters, the dynamics can be on chaotic-nonchaotic and strange-simple attractors. The choice of quasiperiodic forcing is of the form as in Eq. (26). If $f: M\times\mathbb{R}\to M$ is the time evolution of a system in ϕ in an m-dimensional manifold M, the Takens delay embedding map $\mathcal{S}: M\to\mathbb{R}^d$ is given by $\mathcal{S}(\phi)=(s(\phi), s(f(\phi)), \ldots, s(f^{d-1}(\phi)))$ and requires that $d \geqslant 2m+1$. In our case of forced systems the dynamics is on an $M\times N$ manifold where the forcing $g\in N$, an n-dimensional manifold [14].

We take $\mathcal{S}: M\times N\to\mathbb{R}^d$ as the embedding, namely, $\mathcal{S}(\phi,\theta)=(s(\phi,\theta), s(f(\phi,\theta)), \ldots, s(f^{d-1}(\phi,\theta)))$ for $d\geqslant 2(m+n)+1$. This is, however, not generic for all f and g. In addition, periodic forcing of short period needs to be excluded. This is a technical pathology [14] which can be overcome for quasiperiodic forcing at the expense of having to use higher embedding dimensions.

Table I gives the converged Lyapunov exponents computed from the time-series data, i.e., Λ_i^t for $i=1,\ldots,d$ the number of embedding dimensions of the reconstructed phase space. Increasing the embedding dimension does not significantly alter the estimated Lyapunov exponents. Spurious exponents are also estimated by this method as being approximately zero: this is a result of the forced dynamical system having the skew-product structure. We have already shown that our cluster-weighted modeling based algorithm has the capability of reproducing the probability distribution of finite-time exponents. This is true for the SNA dynamics as well, the corresponding probability distribution being asymmetric with distinct domination of contracting dynamics [27]; see Fig. 2. This effect is more prominent for smaller finite-time lengths when it results in longer tails in the distribution. The error margins reported in Table I are the standard deviations of the estimated Lyapunov exponents obtained by repeating the analysis for different initial conditions, namely,

$$\sigma=\sqrt{\frac{1}{m-1}\sum_{k=1}^{m}(\Lambda_k-\langle\Lambda\rangle)^2}, \qquad (27)$$

where m is the number of (Monte Carlo) trials. The distribution of the Λ_k is approximately normal, with mean $\langle\Lambda\rangle\equiv\Lambda^t$.

C. Additive noise

We have also studied the effects of noise in the estimation of Lyapunov exponents. This is a problem of considerable practical importance since any experimental signal is usually corrupted by noise. The usual effect of additive noise is to increase unpredictability, and therefore give higher estimates for the Lyapunov exponent(s). For dynamics that is chaotic, this introduces systematic errors, but when the dynamics is nonchaotic (as on a SNA), additive noise can destroy the nonchaotic attractor. Here we add white noise to the time series of the forced Hénon map as

$$\phi_n \to \phi_n + R\eta_n, \qquad (28)$$

where R is the noise strength and the η_n are i.i.d. random variables in the interval $[-1,1]$. The CWM technique is tolerant to low noise intensities; shown in Fig. 3 are the esti-

TABLE I. Lyapunov exponents estimated from time series (Λ^t) for systems without and with forcing, and on chaotic attractors or SNA's. The largest Lyapunov exponent, denoted Λ^m, is obtained from the dynamics. In each case, a total of 20 Monte Carlo trials was used to estimate the error bars.

System	Model	Parameters	Λ^m	Λ^t
Logistic map	$\phi_{n+1}=\alpha\phi_n(1-\phi_n)$	$\alpha=4.0$ (chaotic)	0.6931	0.6929 ± 0.0049
Forced	$\phi_{n+1}=\alpha\phi_n(1-\phi_n)+\epsilon\sin(2\pi\theta_n)$	$\alpha=0.155, \epsilon=3.04$	0.0161	0.0169 ± 0.0040
logistic map	$\theta_{n+1}=\theta_n+\omega(\text{mod } 1)$	(chaotic)		0.0000 ± 0.0008
				-0.0002 ± 0.0004
		$\alpha=0.151, \epsilon=3.01$	-0.0210	0.0001 ± 0.0004
		(SNA)		0.0000 ± 0.0004
				-0.0212 ± 0.0849
Forced	$\phi_{n+1}=1-\beta\phi_n^2+\psi_n+\alpha\sin(2\pi\theta_n)$	$\alpha=0.2, \gamma=0.1, \beta=1.0$	0.0846	0.0849 ± 0.0062
Hénon map	$\psi_{n+1}=\gamma\phi_n$	(chaotic)		0.0000 ± 0.0008
	$\theta_{n+1}=\theta_n+\omega(\text{mod } 1)$			0.0000 ± 0.0004
				-2.3867 ± 0.0715
		$\alpha=0.2, \gamma=0.1, \beta=0.885$	-0.0377	0.0000 ± 0.0004
		(SNA)		0.0000 ± 0.0008
				-0.0376 ± 0.0053
				-2.2645 ± 0.0093

mated values of the three leading Lyapunov exponents as a function of the noise strength, $\log_{10} R$. For higher noise intensity, the spurious exponents (which should be zero) are inaccurately estimated, leading to an incorrect identification of the dynamics.

V. CONCLUSION

Cluster-weighted modeling is an efficient methodology for the reconstruction of complex nonlinear dynamics. We have shown here that this method offers the possibility of accurately estimating dynamical features such as the Lyapunov exponents from scalar time series monitored from a variety of driven nonlinear dynamical systems.

Our method relies on the Takens embedding theorem, and constructs local nonlinear maps to model the dynamics in the reconstructed phase space. This makes the computation of the Jacobian both efficient and accurate. Embedding the time-series data in higher dimensions introduces spurious exponents but CWM is successful in restricting them to near

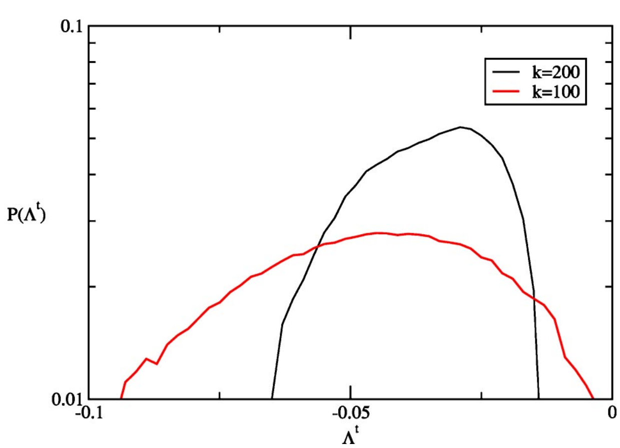

FIG. 2. Characteristic probability distribution of finite-time Lyapunov exponents for the quasiperiodically forced logistic map showing SNA dynamics. The parameters are $\alpha=0.151$, $\epsilon=3.01$, and the distribution is obtained from 2^{12} samples.

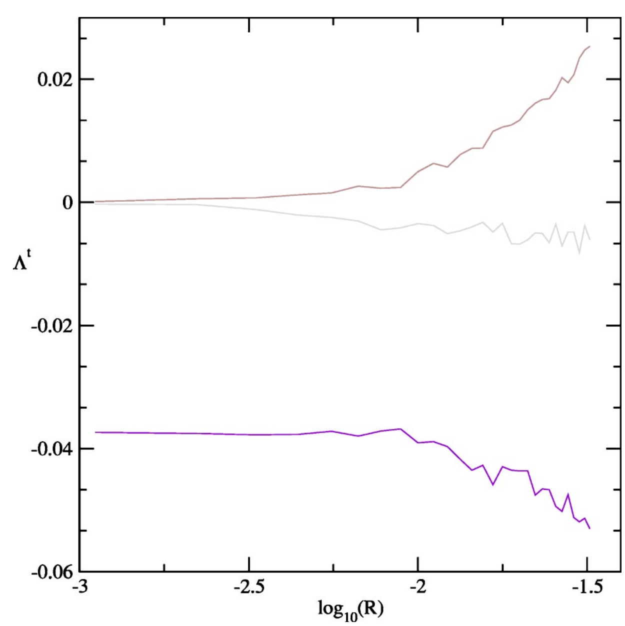

FIG. 3. The effect of external noise R on the leading Lyapunov exponents for the quasiperiodically forced Hénon map showing SNA dynamics. Beyond $R\sim0.003$, accurate estimation of the Lyapunov exponents becomes numerically difficult.

zero values. Of particular importance is the fact that finite-time Lyapunov exponent distributions can also be estimated with considerable accuracy. Finally the present algorithm shows sufficient robustness to noisy data which makes it suitable to be used in most practical situations.

Although strange nonchaotic dynamics have been suggested as underlying stable and aperiodic natural phenomena [28,29], it has hitherto proven difficult to conclusively establish this from time-series analysis alone. The present methods can prove to be of considerable value in studying such dynamics, particularly in an experimental setting.

We also believe that the CWM methodology can be extended to develop control and synchronization strategies for low-dimensional dynamical systems, and work in that direction is currently in progress.

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